Transmission through a quantum dot in a four-terminal phase-coherent system

1998 J. Phys.: Condens. Matter 10 3581

(http://iopscience.iop.org/0953-8984/10/16/011)

J. Phys.: Condens. Matter **10** (1998) 3581–3593. Printed in the UK PII: S0953-8984(98)90653-1

Transmission through a quantum dot in a four-terminal phase-coherent system

Qing-feng Sun† and Tsung-han Lin‡

† Department of Physics, Peking University, Beijing 100871, People's Republic of China
‡ CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China, Institute of Theoretical Physics, Academia Sinica, Beijing 100080, People's Republic of China and Department of Physics, Peking University, Beijing 100871, People's Republic of China§

Received 12 January 1998

Abstract. Motivated by the recent experiment by Schuster *et al* (Schuster R, Buks E, Heiblum M, Mahalu D, Umansky V and Shtrikman H 1997 *Nature* **385** 417), we study a four-terminal phase-coherent system, each arm having a quantum dot embedded in it (dot 1 for studying and dot 0 for reference). Using the nonequilibrium-Green-function method, the open-circuit collector voltage v_4 is derived. We find the following features. (1) The phase behaviours are similar for all of the resonance peaks. (2) In a single resonance peak, the phase φ increases by π on a scale of about the half-peak-width Γ_w. (3) An abrupt phase drop, by π, occurs near the point half way between two consecutive peaks. These results agree well with experiment. We attribute the characteristic (3) to the off-diagonal linewidth of dot 1, which is a single-electron effect. In addition, the crossover of the phase behaviour in going from the four-terminal system to a two-terminal system is studied. Finally, another manifestation of this off-diagonal linewidth is also discussed.

1. Introduction

For very small systems, such as quantum dots, electrons travelling through can maintain their phase coherence. In order to characterize the transport properties fully, it is very important to measure the phase change as an electron passes through such small systems. Yacoby *et al* have measured the phase change of an electron passing through a quantum dot by using a two-terminal phase-coherent set-up [1]. They obtained the following results. (1) There is an abrupt phase increase, by π, on passing a single resonance peak. (2) The transmission amplitudes of the successive resonance peaks are in phase. Because of the limitation of having two terminals, they were not able to observe continuous phase variation. In fact, it is well known that for a two-terminal phase-coherent system, the phase can only take two values (either 0 or π); no continuous phase variation happens. This had been predicted theoretically about ten years ago by Büttiker, on the basis of time-reversal invariance and current conservation [2]. Since the experiment by Yacoby *et al* [1], several theoretical studies have been presented [3–6]. Hackenbroich *et al* explained the abrupt phase increase by π well by treating the intra-dot electron–electron interaction within a self-consistent mean-field approximation [3, 4]. Bruder *et al* investigated nonlinear conductance and considered Kondo-like correlations, and also explained the characteristic (1) for the linear regime [5].

§ Mailing address.

Recently, Schuster *et al* [7] reported the first successful measurement of a continuous phase variation of the electron transmission amplitude through a quantum dot by using a novel experimental set-up, a four-terminal phase-coherent system threaded by a magnetic flux Φ. They found three striking features.

(1) The phase behaviours are similar for all of the resonance peaks.

(2) The phase rises by almost π in a single resonance peak on a scale of about the half-peak-width Γ_w.

(3) A sharp phase drop, by π, occurs near the halfway point between two consecutive peaks on a scale much smaller than Γ_w or $k_B\mathcal{T}$ ($\mathcal{T}$ is the temperature).

Feature (2) is well consistent with the Breit–Wigner formula [8], but feature (3) is in clear contradiction with it. Oreg and Gefen proposed a mechanism in which an inherently finite-temperature many-body effect causes a phase drop [9], but a complete explanation for feature (3) remains to be found.

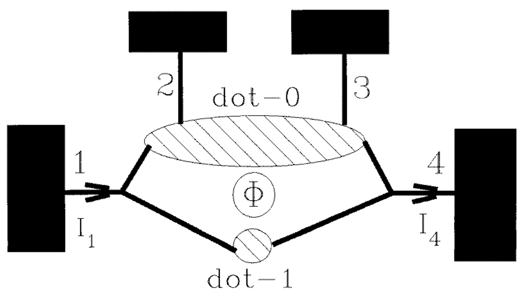

Figure 1. A schematic diagram of the model system: the dark regions represent the reservoirs, dot 1 is coupled to lead 1 and lead 4, dot 0 is coupled to all of the leads, and the system is threaded by a flux Φ.

In this paper our main goal is to explain the results of the experiment of Schuster *et al* [7], in particular, the characteristic (3). We consider a four-terminal phase-coherent system, shown schematically in figure 1, in which each arm has a quantum dot embedded in it (dot 1 for studying and dot 0 for reference), and which is threaded by a magnetic flux Φ. Although the model system under consideration is not the one used in the experiment of reference [7], under certain conditions (see section 2 below), dot 0 approximately plays the role of a wave-guide-like wire, and the model system is close to the experimental situation. By using the nonequilibrium-Green-function method, we derive the collector current I_4. Then, by using the open-circuit condition ($I_4 = 0$), the open-circuit collector voltage v_4 is obtained. The phase behaviour is studied in detail, and we find that it is in good agreement with the experiment of Schuster *et al* [7]. In particular, we can obtain the sharp phase drop near the halfway point between two consecutive peaks, which we attribute to the off-diagonal linewidth of dot 1. It should be emphasized that this mechanism is completely a single-electron effect. In addition, the crossover from a continuous phase increase of the four-terminal system to an abrupt phase increase of the two-terminal system is also studied. Finally, we predict that another manifestation of this off-diagonal linewidth may emerge in a strong-tunnelling situation.

The outline of this paper is as follows. In section 2, we present the model and derive the formula for the collector current I_4 by the Keldysh nonequilibrium-Green-function method. The main results of our theory for the four-terminal model system, including the open-circuit collector voltage v_4 and the phase behaviour, are presented in section 3. The crossover of the phase behaviour in going from the four-terminal system to the two-terminal system is discussed in section 4. Another interesting manifestation of the off-diagonal linewidth is predicted theoretically in section 5. Finally, a brief summary is given in section 6.

2. The model and the collector current

The system under consideration is a four-terminal phase-coherent system with one quantum dot embedded in each arm, and threaded by a magnetic flux Φ. This system can be described by the following Hamiltonian:

$$\begin{aligned} H = & \sum_{k,n} \epsilon_{kn} a_{kn}^{\dagger} a_{kn} + \epsilon_0 b_0^{\dagger} b_0 + \sum_i \epsilon_i c_i^{\dagger} c_i + \sum_{i,j\,(i\neq j)} \frac{U}{2} c_i^{\dagger} c_i c_j^{\dagger} c_j \\ & + \left[\sum_{k,n} w_{kn} a_{kn}^{\dagger} b_0 + \sum_{k,j} v_{kj}^{1} \mathrm{e}^{\mathrm{i}\phi} a_{k1}^{\dagger} c_j + \sum_{k,j} v_{kj}^{4} a_{k4}^{\dagger} c_j + \mathrm{HC} \right] \end{aligned} \tag{1}$$

where $a_{kn}^{\dagger}$ (a_{kn}) is the creation (annihilation) operator for an electron in lead n, $n = 1, 2, 3, 4$ corresponding to lead 1–lead 4, respectively. The second term describes dot 0, in which only a single state is considered. The third and the fourth terms are for dot 1 with multiple energy levels and the Coulomb interaction U between the electrons. The last term describes the tunnelling between the dots and the leads, where lead 2 and lead 3 are only coupled to dot 0 (see figure 1). To account for the system threaded by the magnetic flux Φ, the matrix element connecting dot 1 and lead 1 is set as $v_{kj}^{1}\mathrm{e}^{\mathrm{i}\phi}$ [3, 4], where $\phi = 2\pi\Phi/\Phi_0$ and v_{kj}^{1} is the matrix element without the magnetic field.

The collector current flowing from the system into lead 4 can be calculated from the evolution of the total number operator of the electrons in lead 4 [10, 11],

$$N_4 = \sum_k a_{k4}^{\dagger} a_{k4}.$$

Then one finds

$$I_4 = e\langle \dot{N}_4 \rangle = -\mathrm{i}e\langle [N_4, H] \rangle = -2e\,\mathrm{Re} \sum_k w_{k4} G_{0,k4}^{<}(t,t) - 2e\,\mathrm{Re} \sum_{k,i} v_{ki}^{4} G_{i,k4}^{<}(t,t) \tag{2}$$

where the Green functions $G_{0,k4}^{<}(t,t')$ and $G_{i,k4}^{<}(t,t')$ are defined as

$$G_{0,k4}^{<}(t,t') \equiv \mathrm{i}\langle a_{k4}^{\dagger}(t') b_0(t) \rangle \qquad G_{i,k4}^{<}(t,t') \equiv \mathrm{i}\langle a_{k4}^{\dagger}(t') c_i(t) \rangle.$$

With the help of the Dyson equation, the Green function $G_{\alpha,k4}^{<}(t,t')$ ($\alpha = 0, i$) can be expressed as

$$\begin{aligned} G_{\alpha,k4}^{<}(t,t') = \int \mathrm{d}t_1 \Bigg\{ & w_{k4}^{*} \left[G_{\alpha 0}^{r}(t,t_1) g_{k4}^{<}(t_1,t') + G_{\alpha 0}^{<}(t,t_1) g_{k4}^{a}(t_1,t') \right] \\ & + \sum_j v_{kj}^{4*} \left[G_{\alpha j}^{r}(t,t_1) g_{k4}^{<}(t_1,t') + G_{\alpha j}^{<}(t,t_1) g_{k4}^{a}(t_1,t') \right] \Bigg\} \end{aligned} \tag{3}$$

where $g_{k4}^{<}$, g_{k4}^{a} are the exact Green functions of the electron in lead 4 without coupling between the leads and the dots; the Green functions $G_{\alpha\beta}^{r}(t,t_1)$ and $G_{\alpha\beta}^{<}(t,t_1)$ ($\alpha = 0$, i; $\beta = 0$, j) are defined as

$$\begin{pmatrix} G_{00}^{r}(t,t_1) & G_{0j}^{r}(t,t_1) \\ G_{i0}^{r}(t,t_1) & G_{ij}^{r}(t,t_1) \end{pmatrix} \equiv -\mathrm{i}\theta(t-t_1) \begin{pmatrix} \langle\{b_0(t), b_0^{\dagger}(t_1)\}\rangle & \langle\{b_0(t), c_j^{\dagger}(t_1)\}\rangle \\ \langle\{c_i(t), b_0^{\dagger}(t_1)\}\rangle & \langle\{c_i(t), c_j^{\dagger}(t_1)\}\rangle \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} G_{00}^{<}(t,t_1) & G_{0j}^{<}(t,t_1) \\ G_{i0}^{<}(t,t_1) & G_{ij}^{<}(t,t_1) \end{pmatrix} \equiv \mathrm{i} \begin{pmatrix} \langle b_0^{\dagger}(t_1) b_0(t) \rangle & \langle c_j^{\dagger}(t_1) b_0(t) \rangle \\ \langle b_0^{\dagger}(t_1) c_i(t) \rangle & \langle c_j^{\dagger}(t_1) c_i(t) \rangle \end{pmatrix}. \tag{5}$$

Substituting the expressions for $G^<_{0,k4}(t,t)$ and $G^<_{i,k4}(t,t)$ into (2), the sum over k can be changed into an integral, $\int d\epsilon\ \rho_4(\epsilon)$, where $\rho_4(\epsilon) = \sum_k \delta(\epsilon - \epsilon_{k4})$ is the density of states of lead 4. Then the collector current I_4 becomes

$$
\begin{aligned}
I_4 = 2e\,\mathrm{Im}\int \frac{d\epsilon}{2\pi}\ &\Bigg\{ f_4(\epsilon)\Bigg[\Gamma^4_0 G^r_{00}(\epsilon) + \sum_j \Gamma^4_{0j} G^r_{j0}(\epsilon) + \sum_i \Gamma^4_{i0} G^r_{0i}(\epsilon) + \sum_{i,j} \Gamma^4_{ij} G^r_{ji}(\epsilon)\Bigg] \\
&+ \frac{1}{2}\Bigg[\Gamma^4_0 G^<_{00}(\epsilon) + \sum_j \Gamma^4_{0j} G^<_{j0}(\epsilon) + \sum_i \Gamma^4_{i0} G^<_{0i}(\epsilon) + \sum_{i,j} \Gamma^4_{ij} G^<_{ji}(\epsilon)\Bigg]\Bigg\} \\
\equiv 2e\,\mathrm{Im}\int \frac{d\epsilon}{2\pi}\ &\Bigg\{ f_4(\epsilon)\,\mathrm{Tr}\big[\mathbf{\Gamma}^4(\epsilon)\mathbf{G}^r(\epsilon)\big] + \frac{1}{2}\mathrm{Tr}\big[\mathbf{\Gamma}^4(\epsilon)\mathbf{G}^<(\epsilon)\big]\Bigg\}
\end{aligned} \tag{6}
$$

in which $f_4(\epsilon)$ is the Fermi distribution function of the electrons in lead 4, and $\mathbf{\Gamma}^4(\epsilon)$ is a matrix linewidth function defined as

$$
\begin{aligned}
\mathbf{\Gamma}^4(\epsilon) = \begin{pmatrix} \Gamma^4_0 & \Gamma^4_{0j} \\ \Gamma^4_{i0} & \Gamma^4_{ij} \end{pmatrix} &= \sum_k 2\pi\delta(\epsilon - \epsilon_{k4}) \begin{pmatrix} w^*_{k4} w_{k4} & w^*_{k4} v^4_{kj} \\ v^{4*}_{ki} w_{k4} & v^{4*}_{ki} v^4_{kj} \end{pmatrix} \\
&= 2\pi\rho_4(\epsilon) \begin{pmatrix} w^*_4(\epsilon) w_4(\epsilon) & w^*_4(\epsilon) v^4_j(\epsilon) \\ v^{4*}_i(\epsilon) w_4(\epsilon) & v^{4*}_i(\epsilon) v^4_j(\epsilon) \end{pmatrix}.
\end{aligned} \tag{7}
$$

It should be stressed that in this work we will retain the off-diagonal linewidths

$$
\Gamma^n_{ij} \equiv \sum_k 2\pi\delta(\epsilon - \epsilon_{kn}) v^{n*}_{ki} v^n_{kj} \qquad (n = 1, 4)
$$

which are usually neglected, as in reference [12] and [13]. It turns out that these off-diagonal linewidths of the quantum dot play an essential role in producing the sharp phase drop mentioned above. In equation (6), the matrix Green function $\mathbf{G}^\alpha(\epsilon)$ $(\alpha = r, <)$ is the Fourier transform of the matrix Green function $\mathbf{G}^\alpha(t, 0)$ defined by

$$
\begin{aligned}
\mathbf{G}^\alpha(\epsilon) = \begin{pmatrix} G^\alpha_{00}(\epsilon) & G^\alpha_{0j}(\epsilon) \\ G^\alpha_{i0}(\epsilon) & G^\alpha_{ij}(\epsilon) \end{pmatrix} &= \int dt\ e^{i\epsilon t} \begin{pmatrix} G^\alpha_{00}(t,0) & G^\alpha_{0j}(t,0) \\ G^\alpha_{i0}(t,0) & G^\alpha_{ij}(t,0) \end{pmatrix} \\
&\equiv \begin{pmatrix} \langle\langle b_0|b^\dagger_0\rangle\rangle^\alpha & \langle\langle b_0|c^\dagger_j\rangle\rangle^\alpha \\ \langle\langle c_i|b^\dagger_0\rangle\rangle^\alpha & \langle\langle c_i|c^\dagger_j\rangle\rangle^\alpha \end{pmatrix}.
\end{aligned} \tag{8}
$$

In the last line of equation (8), the Green functions are expressed in the forms $\langle\langle X|Y\rangle\rangle^\alpha$ $(\alpha = r, <)$, where X, Y denote b_0 or c_j. These forms will be convenient for the following calculation.

In order to derive the collector current I_4, we have to calculate two traces: $\mathrm{Tr}[\mathbf{\Gamma}^4\mathbf{G}^r]$ and $\mathrm{Tr}[\mathbf{\Gamma}^4\mathbf{G}^<]$. First, let us calculate $\mathrm{Tr}[\mathbf{\Gamma}^4\mathbf{G}^r]$. By using the equation of motion (EOM) $\epsilon\langle\langle X|Y\rangle\rangle^r = \langle\langle [X, H]|Y\rangle\rangle^r + \langle\{X, Y\}\rangle$, we have

$$
(\epsilon - \epsilon_0)\langle\langle b_0|b^\dagger_0\rangle\rangle^r = 1 + \sum_{k,n} w^*_{kn}\langle\langle a_{kn}|b^\dagger_0\rangle\rangle^r \tag{9}
$$

$$
(\epsilon - \epsilon_i)\langle\langle c_i|b^\dagger_0\rangle\rangle^r = U\sum_{j\,(j\neq i)}\langle\langle c_i c^\dagger_j c_j|b^\dagger_0\rangle\rangle^r + \sum_k v^{1*}_{ki} e^{-i\phi}\langle\langle a_{k1}|b^\dagger_0\rangle\rangle^r + \sum_k v^{4*}_{ki}\langle\langle a_{k4}|b^\dagger_0\rangle\rangle^r. \tag{10}
$$

For the closure of the EOM, the higher-order two-particle Green function $\langle\langle c_i c^\dagger_j c_j|b^\dagger_0\rangle\rangle^r$ must be decoupled. We make the following decoupling approximation [14]:

$$
\langle\langle c_i c^\dagger_j c_j|b^\dagger_0\rangle\rangle^r = N_j\langle\langle c_i|b^\dagger_0\rangle\rangle^r \tag{11}
$$

where N_j is the occupation number of state j of dot 1. This decoupling scheme is equivalent to the mean-field approximation, and the only effect of the electron–electron interaction is to separate the neighbouring resonances by a spacing of U/e [3, 4]. The new retarded Green functions $\langle\langle a_{kn}|b_0^\dagger\rangle\rangle^r$ ($n = 1$, 2, 3, and 4) in equations (9) and (10) can be obtained from Dyson's equation:

$$\begin{aligned}
\langle\langle a_{k1}|b_0^\dagger\rangle\rangle^r &= \langle\langle a_{k1}|a_{k1}^\dagger\rangle\rangle_0^r \left\{ w_{k1}\langle\langle b_0|b_0^\dagger\rangle\rangle^r + \sum_j v_{kj}^1 \mathrm{e}^{\mathrm{i}\phi}\langle\langle c_j|b_0^\dagger\rangle\rangle^r \right\} \\
\langle\langle a_{kn}|b_0^\dagger\rangle\rangle^r &= \langle\langle a_{kn}|a_{kn}^\dagger\rangle\rangle_0^r w_{kn}\langle\langle b_0|b_0^\dagger\rangle\rangle^r \qquad \text{for } n = 2, 3 \\
\langle\langle a_{k4}|b_0^\dagger\rangle\rangle^r &= \langle\langle a_{k4}|a_{k4}^\dagger\rangle\rangle_0^r \left\{ w_{k4}\langle\langle b_0|b_0^\dagger\rangle\rangle^r + \sum_i v_{ki}^4 \langle\langle c_i|b_0^\dagger\rangle\rangle^r \right\}
\end{aligned} \tag{12}$$

where $\langle\langle a_{kn}|a_{kn}^\dagger\rangle\rangle_0^r = 1/(\epsilon - \epsilon_{kn} + \mathrm{i}0^+)$ ($n = 1$, 2, 3, and 4) are the exact retarded Green functions in lead n without coupling between the leads and the dots. We substitute the expressions for $\langle\langle a_{k4}|b_0^\dagger\rangle\rangle^r$ into equations (9) and (10), and, as in most of the literature, make two further simplifications.

(1) We make the wide-bandwidth approximation [15], i.e. all of the linewidths (Γ_0^n, $\mathbf{\Gamma}^1$, and $\mathbf{\Gamma}^4$) are treated as constants, independent of ϵ. Then one has

$$\sum_k w_{kn}^* w_{kn}\langle\langle a_{kn}|a_{kn}^\dagger\rangle\rangle_0^r = -\frac{\mathrm{i}}{2}\Gamma_0^n$$

where

$$\Gamma_0^n \equiv \sum_k 2\pi\delta(\epsilon - \epsilon_{kn}) w_{kn}^* w_{kn}.$$

(2) We let the left-hand and right-hand barriers be symmetric, i.e. let $\mathbf{\Gamma}^1 = \mathbf{\Gamma}^4 \equiv \frac{1}{2}\mathbf{\Gamma}$. Then equations (9) and (10) become

$$\left(\epsilon - \epsilon_0 + \frac{\mathrm{i}}{2}\sum_n \Gamma_0^n\right)\langle\langle b_0|b_0^\dagger\rangle\rangle^r = 1 + A\sum_i \Gamma_{0i}^4\langle\langle c_i|b_0^\dagger\rangle\rangle^r \tag{13}$$

$$(\epsilon - \epsilon_i - UN_i')\langle\langle c_i|b_0^\dagger\rangle\rangle^r = -A^*\Gamma_{i0}^4\langle\langle b_0|b_0^\dagger\rangle\rangle^r - \mathrm{i}\sum_j \Gamma_{ij}\langle\langle c_j|b_0^\dagger\rangle\rangle^r. \tag{14}$$

Here

$$A = -\frac{\mathrm{i}}{2}(1 + \mathrm{e}^{\mathrm{i}\phi}) \qquad N_i' = \sum_{j(j\neq i)} N_j.$$

From equation (14), one easily finds that

$$\sum_i \Gamma_{0i}^4\langle\langle c_i|b_0^\dagger\rangle\rangle^r = -A^*\sum_i \frac{\Gamma_{0i}^4\Gamma_{i0}^4}{\epsilon - \epsilon_i - UN_i'}\langle\langle b_0|b_0^\dagger\rangle\rangle^r - \mathrm{i}\sum_{ij}\frac{\Gamma_{0i}^4\Gamma_{ij}^4}{\epsilon - \epsilon_i - UN_i'}\langle\langle c_j|b_0^\dagger\rangle\rangle^r. \tag{15}$$

Notice that $\Gamma_{0i}^4\Gamma_{i0}^4 = \Gamma_0^4\Gamma_{ii}^4$ and $\Gamma_{0i}^4\Gamma_{ij}^4 = \Gamma_{0j}^4\Gamma_{ii}^4$, so $\sum_i \Gamma_{0i}^4\langle\langle c_i|b_0^\dagger\rangle\rangle^r$ can be obtained as

$$\sum_i \Gamma_{0i}^4\langle\langle c_i|b_0^\dagger\rangle\rangle^r = -A^* B\Gamma_0^4\langle\langle b_0|b_0^\dagger\rangle\rangle^r \tag{16}$$

where

$$B(\epsilon) \equiv (1/2)\Bigg/\left[\left(\sum_i \frac{\Gamma_{ii}}{\epsilon - \epsilon_i - UN_i'}\right)^{-1} + \mathrm{i}/2\right]. \tag{17}$$

In fact $B(\epsilon)$ is just the amplitude of transmission through dot 1 with the off-diagonal linewidth Γ_{ij} taken into consideration. On combining equations (16) and (13), $\langle\langle b_0|b_0^\dagger\rangle\rangle^r$ is obtained straightforwardly:

$$\langle\langle b_0|b_0^\dagger\rangle\rangle^r = 1\Bigg/\left(\epsilon-\epsilon_0+\frac{\mathrm{i}}{2}\sum_n \Gamma_0^n + |A|^2 B\Gamma_0^4\right) \tag{18}$$

Now let us calculate $\sum_i \Gamma_{i0}^4\langle\langle b_0|c_i^\dagger\rangle\rangle^r$ and $\sum_{ij}\Gamma_{ij}^4\langle\langle c_j|c_i^\dagger\rangle\rangle^r$. By using the equation of motion $-\epsilon\langle\langle X|Y\rangle\rangle^r = \langle\langle X|[Y,H]\rangle\rangle^r - \langle\{X,Y\}\rangle$, and the Dyson equation, one finds that

$$(\epsilon-\epsilon_i-UN_i')\langle\langle b_0|c_i^\dagger\rangle\rangle^r = \sum_k v_{ki}^1 \mathrm{e}^{\mathrm{i}\phi}\langle\langle b_0|a_{k1}^\dagger\rangle\rangle^r + \sum_k v_{ki}^4\langle\langle b_0|a_{k4}^\dagger\rangle\rangle^r \tag{19}$$

$$(\epsilon-\epsilon_i-UN_i')\langle\langle c_j|c_i^\dagger\rangle\rangle^r = \delta_{ij} + \sum_k v_{ki}^1 \mathrm{e}^{\mathrm{i}\phi}\langle\langle c_j|a_{k1}^\dagger\rangle\rangle^r + \sum_k v_{ki}^4\langle\langle c_j|a_{k4}^\dagger\rangle\rangle^r \tag{20}$$

$$\langle\langle X|a_{k1}^\dagger\rangle\rangle^r = \langle\langle a_{k1}|a_{k1}^\dagger\rangle\rangle_0^r\left\{w_{k1}^*\langle\langle X|b_0^\dagger\rangle\rangle^r + \sum_j v_{kj}^{1*}\mathrm{e}^{-\mathrm{i}\phi}\langle\langle X|c_j^\dagger\rangle\rangle^r\right\} \tag{21}$$

$$\langle\langle X|a_{k4}^\dagger\rangle\rangle^r = \langle\langle a_{k4}|a_{k4}^\dagger\rangle\rangle_0^r\left\{w_{k4}^*\langle\langle X|b_0^\dagger\rangle\rangle^r + \sum_j v_{kj}^{4*}\langle\langle X|c_j^\dagger\rangle\rangle^r\right\}. \tag{22}$$

X, Y in equations (21) and (22) denote b_0 or c_j, Substituting equations (21) and (22) into equations (19) and (20), we have

$$(\epsilon-\epsilon_i-UN_i')\langle\langle b_0|c_i^\dagger\rangle\rangle^r = A\Gamma_{0i}^4\langle\langle b_0|b_0^\dagger\rangle\rangle^r - \mathrm{i}\sum_j \Gamma_{ji}^4\langle\langle b_0|c_j^\dagger\rangle\rangle^r \tag{23}$$

$$(\epsilon-\epsilon_i-UN_i')\langle\langle c_j|c_i^\dagger\rangle\rangle^r = \delta_{ij} + A\Gamma_{0i}^4\langle\langle c_j|b_0^\dagger\rangle\rangle^r - \mathrm{i}\sum_l \Gamma_{li}^4\langle\langle c_j|c_l^\dagger\rangle\rangle^r. \tag{24}$$

After some algebraic manipulations, and noticing that: (1) $\Gamma_{i0}^4\Gamma_{0i}^4 = \Gamma_0^4\Gamma_{ii}^4$; (2) $\Gamma_{ji}^4\Gamma_{i0}^4 = \Gamma_{j0}^4\Gamma_{ii}^4$; (3) $\Gamma_{ij}^4\Gamma_{li}^4 = \Gamma_{ii}^4\Gamma_{lj}^4$, we obtain

$$\sum_i \Gamma_{i0}^4\langle\langle b_0|c_i^\dagger\rangle\rangle^r = AB\Gamma_0^4\langle\langle b_0|b_0^\dagger\rangle\rangle^r \tag{25}$$

and

$$\sum_{ij}\Gamma_{ij}^4\langle\langle c_j|c_i^\dagger\rangle\rangle^r = \left(\epsilon_0-\epsilon_0+\frac{\mathrm{i}}{2}\sum_n\Gamma_0^n\right)B\langle\langle b_0|b_0^\dagger\rangle\rangle^r. \tag{26}$$

By combining equations (16), (18), (25), and (26), one finally obtains the trace $\mathrm{Tr}[\boldsymbol{\Gamma}^4\mathbf{G}^r]$ as

$$\begin{aligned}\mathrm{Tr}[\boldsymbol{\Gamma}^4\mathbf{G}^r(\epsilon)] &= \Gamma_0^4\langle\langle b_0|b_0^\dagger\rangle\rangle^r + \sum_i \Gamma_{0i}^4\langle\langle c_i|b_0^\dagger\rangle\rangle^r + \sum_i\Gamma_{i0}^4\langle\langle b_0|c_i^\dagger\rangle\rangle^r + \sum_{ij}\Gamma_{ij}^4\langle\langle c_j|c_i^\dagger\rangle\rangle^r \\ &= \left[\Gamma_0^4 + (A-A^*)B\Gamma_0^4 + \left(\epsilon-\epsilon_0+\frac{\mathrm{i}}{2}\sum_n\Gamma_0^n\right)B\right]G_{00}^r.\end{aligned} \tag{27}$$

The next step is to calculate the trace $\mathrm{Tr}[\boldsymbol{\Gamma}^4\mathbf{G}^<]$. We use the Keldysh equation $\mathbf{G}^< = \mathbf{G}^r\boldsymbol{\Sigma}^<\mathbf{G}^a$, where $\mathbf{G}^a$ is the advanced Green function and $\boldsymbol{\Sigma}^<$ is the self-energy, which can be easily obtained under the wide-bandwidth approximation:

$$\begin{aligned}\boldsymbol{\Sigma}^<(\epsilon) &= \mathrm{i}f_1(\epsilon)\begin{pmatrix}\Gamma_0^1 & \Gamma_{0i}^1\mathrm{e}^{\mathrm{i}\phi}\\ \Gamma_{j0}^1\mathrm{e}^{-\mathrm{i}\phi} & \Gamma_{ji}^1\end{pmatrix} + \mathrm{i}f_2(\epsilon)\begin{pmatrix}\Gamma_0^2 & 0\\ 0 & 0\end{pmatrix} + \mathrm{i}f_3(\epsilon)\begin{pmatrix}\Gamma_0^3 & 0\\ 0 & 0\end{pmatrix}\\ &\quad + \mathrm{i}f_4(\epsilon)\begin{pmatrix}\Gamma_0^4 & \Gamma_{0i}^4\\ \Gamma_{j0}^4 & \Gamma_{ji}^4\end{pmatrix}.\end{aligned} \tag{28}$$

Substituting the expression for the self-energy $\boldsymbol{\Sigma}^<$ into $\mathbf{G}^<$, one immediately sees that the trace $\mathrm{Tr}[\boldsymbol{\Gamma}^4\mathbf{G}^<]$ is a linear function of $\mathrm{i}f_n(\epsilon)$ ($n = 1$, 2, 3, and 4). Noticing that Γ_0^4, Γ_{0j}^4, Γ_{i0}^4, and Γ_{ij}^4 obey the above-mentioned relations, the coefficients of $\mathrm{i}f_n(\epsilon)$ can be calculated one by one, and the trace $\mathrm{Tr}[\boldsymbol{\Gamma}^4\mathbf{G}^<]$ is obtained:

$$\begin{aligned}\mathrm{Tr}[\boldsymbol{\Gamma}^4\mathbf{G}^<] = {} & \mathrm{i}f_1(\epsilon)\left|\Gamma_0^4 - 2A^*B\Gamma_0^4 + \left(\epsilon-\epsilon_0+\frac{\mathrm{i}}{2}\sum_n\Gamma_0^n\right)B\mathrm{e}^{-\mathrm{i}\phi}\right|^2 \left|G_{00}^r\right|^2 \\ & + \mathrm{i}f_2(\epsilon)\Gamma_0^2\Gamma_0^4|1-A^*B|^2\left|G_{00}^r\right|^2 + \mathrm{i}f_3(\epsilon)\Gamma_0^3\Gamma_0^4|1-A^*B|^2\left|G_{00}^r\right|^2 \\ & + \mathrm{i}f_4(\epsilon)\left|\Gamma_0^4 + (A-A^*)B\Gamma_0^4 + \left(\epsilon-\epsilon_0+\frac{\mathrm{i}}{2}\sum_n\Gamma_0^n\right)B\right|^2\left|G_{00}^r\right|^2. \end{aligned} \tag{29}$$

Finally, substituting these two traces, $\mathrm{Tr}[\boldsymbol{\Gamma}^4\mathbf{G}^r]$ and $\mathrm{Tr}[\boldsymbol{\Gamma}^4\mathbf{G}^<]$, into equation (6), and noticing that $B^* - B = 2\mathrm{i}|B|^2$ and $\Gamma_{ij}^4\Gamma_{lm}^4 = \Gamma_{im}^4\Gamma_{lj}^4$, the formula for the collector current I_4 can be expressed as follows:

$$I_4 = e\int\frac{\mathrm{d}\epsilon}{2\pi}\left[\sum_{i=1,2,3}T_{i4}(f_i - f_4)\right]. \tag{30}$$

Formally, equation (30) is the multiple-probe Büttiker formula, where $T_{i4}(\epsilon)$ ($i = 1, 2, 3$) is the transmission probability including all of the phase information:

$$\begin{aligned} T_{14} &= \left|\Gamma_0^4 - 2A^*B\Gamma_0^4 + \left(\epsilon-\epsilon_0+\frac{\mathrm{i}}{2}\sum_n\Gamma_0^n\right)B\mathrm{e}^{-\mathrm{i}\phi}\right|^2\left|G_{00}^r\right|^2 \\ T_{i4} &= \Gamma_0^i\Gamma_0^4|1-A^*B|^2\left|G_{00}^r\right|^2 \qquad (i = 2, 3). \end{aligned} \tag{31}$$

It should be emphasized that we have considered all of the linewidths of dot 1, including the diagonal linewidth Γ_{ii} and the off-diagonal linewidths Γ_{ij} which are usually neglected, as in the previous studies [12, 13]. Since the Γ_{ij} are not independent of one another, they satisfy $\Gamma_{ij}\Gamma_{lm} = \Gamma_{im}\Gamma_{lj}$, $\Gamma_{ij} = \Gamma_{ji}^*$, $|\Gamma_{ij}|^2 = \Gamma_{ii}\Gamma_{jj}$, so only the diagonal linewidths Γ_{ii} appear in equations (30), (31), and in the expression for the transmission amplitude through dot 1, B (see equation (17)).

The occupation number for the state i of dot 1, N_i, should usually be calculated self-consistently. But for simplicity, here we neglect the coupling with dot 0, and thus we have

$$N_i = \int\frac{\mathrm{d}\epsilon}{2\pi}\,\frac{f_1\Gamma_{ii}^1 + f_4\Gamma_{ii}^4}{(\epsilon-\epsilon_i-UN_i')^2 + (\Gamma_{ii})^2/4}. \tag{32}$$

Equation (30) is the central formula of this work.

3. The main results and the comparison with experiment

In this section, we will study the phase behaviour and some other properties of the model four-terminal system. To imitate the experiment of Schuster *et al* [7], we choose our parameters with (a) $\min(\Gamma_0^1, \Gamma_0^2, \Gamma_0^3, \Gamma_0^4) \gg \max(\Gamma_1^1, \Gamma_1^4, k_B\mathcal{T})$ and (b) ϵ_0 far from the chemical potential μ, i.e. $|\epsilon_0 - \mu_i| \gg \max(\Gamma_1^1, \Gamma_1^4, k_B\mathcal{T})$. Under these conditions, the amplitude of transmission through dot 0 is approximately a constant over a range of several $k_B\mathcal{T}$, and Γ_1^4 is around μ. Therefore, dot 0 in our model can be approximately considered as a wave-guide-like wire, and the system is reduced to a four-terminal system with only one dot (dot 1) in an arm. Also, (c) we let $\mu_2 = \mu_3 = 0$, to describe the connections of lead 2 and lead 3 to the base.

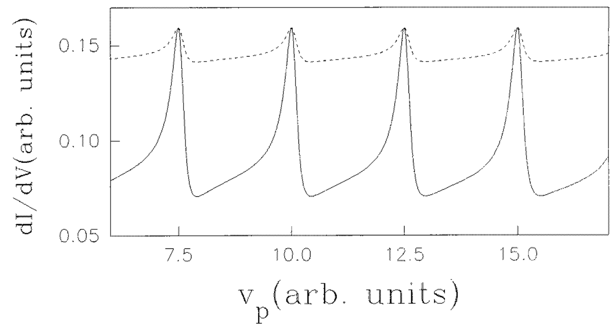

Figure 2. The emitter conductance dI_1/dv_1 and the collector conductance dI_4/dv_1 versus v_p for $\epsilon_i = -v_p + (i-1)\Delta\epsilon$ and $\Phi = 0$. The parameters chosen are: $\mathcal{T} = 0$, $\Gamma_0^1 = \Gamma_0^4 = 500$, $\Gamma_0^2 = \Gamma_0^3 = 200$, $\Gamma_{ii}^1 = \Gamma_{ii}^4 = 0.5$, $\epsilon_0 = 100$, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$. We assume that dot 1 has ten states with $\Delta\epsilon = 2.5$ and $U = 0$. The resonance peaks from the fourth to the seventh are shown in the figure.

3.1. The emitter conductance and the collector conductance

On the basis of equation (30), we first calculate the dependence of the emitter conductance dI_1/dv_1 ($dI_1/dv_1 = -dI_4/dv_4$ due to the symmetry of the system) and the collector conductance dI_4/dv_1 on the gate voltage v_p applied to dot 1 for $\mu_1 = \mu_4 = 0$ (shown in figure 2). All of the numerical calculations are performed in units for which $\hbar = e = 1$.

In figure 2, a series of the resonance peaks appear. Notice that the magnitudes of dI_1/dv_1 and dI_4/dv_1 are exactly the same at complete resonance, which means that all of the electrons emitted from lead 1 will flow into lead 4, due to the fact that when dot 1 is in resonance, its transmission probability is 1 and the resistance is zero.

3.2. The open-circuit collector voltage

Now we consider the open-circuit case. Let $I_4 = 0$ in equation (30); the open-circuit collector voltage v_4 can be obtained in the linear response regime as

$$\frac{v_4}{v_1} = \left(\int d\epsilon\ T_{14}(\partial f/\partial\epsilon)\right) \Big/ \left(\int d\epsilon \left[\sum_{i=1,2,3} T_{i4}\right](\partial f/\partial\epsilon)\right). \tag{33}$$

From equation (33) one finds that the open-circuit collector voltage, v_4, is smaller than the emitter voltage v_1 but larger than the base voltages without any restriction on the temperature $\mathcal{T}$, the magnetic flux Φ, the gate voltage v_p, the linewidths Γ, and the parameters of the dots (ϵ_0, ϵ_i, and U). This means that the voltage of the open-circuit collector, v_4, is neither the highest nor the lowest among the four voltages of the leads; otherwise the open-circuit condition $I_4 = 0$ cannot be satisfied. This is quite reasonable physically.

The curves for v_4/v_1 versus the gate voltage v_p for Γ_{ii} independent of the state i are shown in figure 3(a); they exhibit a series of the resonance peaks on a large background contributed from the reference path. These peaks are slightly asymmetric, and the maximum value of v_4/v_1 is 1 at complete resonance. If finite temperature is considered, the resonance

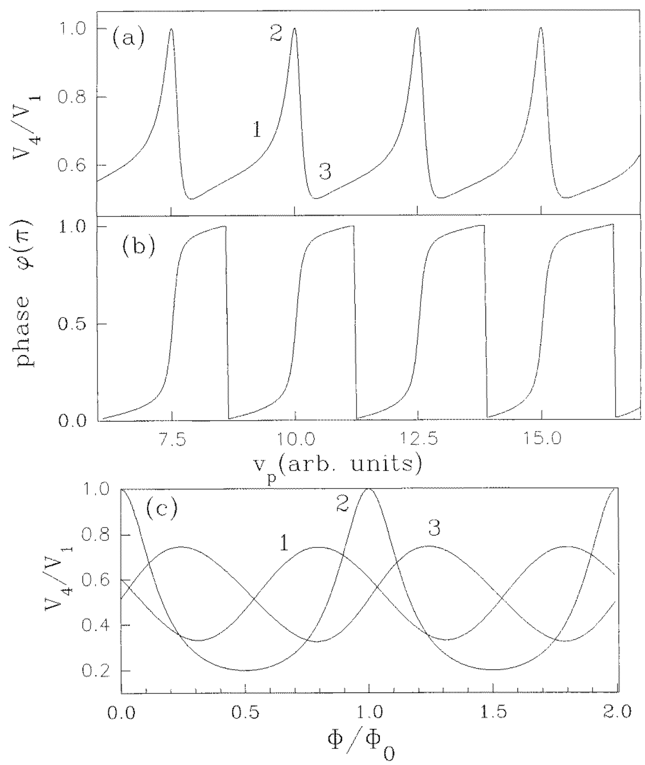

Figure 3. (a) v_4/v_1 versus v_p for $\Phi = 0$; (b) the phase φ versus v_p. For (a) and (b), the parameters are the same as for figure 2. (c) v_4/v_1 versus Φ from point 1 to point 3 in (a), corresponding to $v_p = 9.2$, 10, 10.7, respectively.

peaks will be broadened and lowered. Notice that the half-peak-width Γ_w is not equal to the linewidth of dot 1, Γ_{ii}. In fact, the coupling of dot 1 and the reference arm (dot 0) causes the half-peak-width to be such that $\Gamma_w < \Gamma_{ii}$. The dependence of v_4/v_1 on the magnetic flux Φ at fixed gate voltage v_p is shown in figure 3(c); it exhibits periodic oscillations.

3.3. The phase behaviour

Now let us focus on the phase variation. With the increase of the magnetic flux Φ, the open-circuit collector voltage v_4 exhibits periodic oscillations (see figure 3(c)). The phase of the lowest-order harmonic wave, $\varphi_0(v_p)$, can be easily calculated from the expressions

$$\cos\varphi_0 \propto \int_0^{2\pi} \mathrm{d}\phi\ (\cos\phi)v_4/v_1$$

and

$$\sin\varphi_0 \propto \int_0^{2\pi} \mathrm{d}\phi\ (\sin\phi)v_4/v_1.$$

Then the phase shift φ through the dot 1 is $\varphi = \varphi_0(v_p) - \varphi_0(-\infty)$; here the phase shift φ at $v_p = -\infty$ is set as 0. Figure 3(b) shows the phase φ versus the gate voltage v_p in the case where Γ_{ii} is independent of the state i and temperature $\mathcal{T} = 0$. The properties of the phase variation obtained in this work are as follows.

(1) The phase behaviour is similar for all of the resonance peaks.

(2) In a single resonance peak the phase increases continuously, by a total of π, on a scale of about the half-peak-width Γ_w; this is very different from the situation for the two-terminal phase-coherent system [2–6].

(3) An abrupt phase drop, by π, occurs near the halfway point between two consecutive resonance peaks.

The change is completely abrupt at temperature $\mathcal{T} = 0$, i.e. in the zero-energy regime. We attribute this abrupt phase drop to the off-diagonal linewidths Γ_{ij} of dot 1, which reflect an indirect coupling between different states of dot 1 through the tunnelling between dot 1 and the leads. If we neglect the off-diagonal linewidth, the transport modes through the different states of dot 1 are independent, and the amplitude of transmission through dot 1 is simply a sum of the displaced Breit–Wigner amplitudes as used in reference [7]; the phase drop by π will happen on an energy scale of Γ_{ii}.

Moreover, the magnitude of the oscillation of the lowest-order harmonic wave versus the gate voltage v_p appears as a series of peaks, wider than the resonance peaks (not shown here, but easily understood from figure 2(c)). The magnitude is zero at the abrupt-drop point of the phase variation.

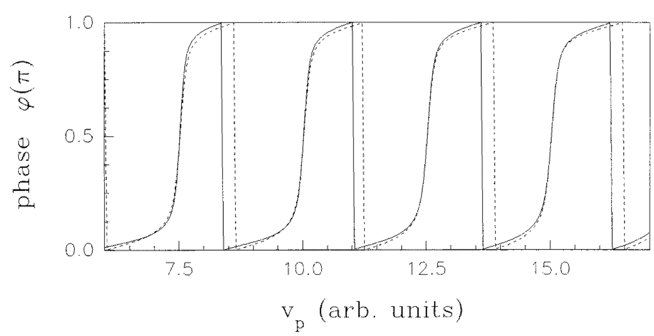

Figure 4. The phase φ versus v_p for Γ_{ii} dependent on the state i, obtained by setting $\Gamma^1_{11} = \Gamma^4_{11} = 0.3$, $\Gamma^1_{ii} = \Gamma^4_{ii} = 1.1\Gamma^4_{i-1,i-1}$. The other parameters are the same as for figure 2. The dotted curve shows the case where Γ_{ii} is independent of the state i ($\Gamma^1_{ii} = \Gamma^4_{ii} = 0.5$), for comparison.

All of the above-mentioned results are well consistent with the experiment of Schuster *et al* [7]. In particular, the steep phase drop is explained. Notice that if Γ_{ii} depends on the state i and the temperature $\mathcal{T}$ is not zero, the properties of the phase variation will undergo no qualitative change. Figure 4 shows the phase φ versus the gate voltage v_p for Γ_{ii} dependent on the state i. The abrupt phase drop, by π, still remains, but the location of the abrupt-drop point will be slightly shifted, as determined by the equation

$$\sum_j \Gamma_{jj}/(\mu_1 - \epsilon_j) = 0.$$

Figure 5 shows the phase φ versus the gate voltage v_p for finite temperature ($\mathcal{T} \neq 0$). In this case the phase drop of about π is not completely abrupt, but a rather sharp drop of the phase still exists near the halfway point between two consecutive resonance peaks, on an energy scale much smaller than both the half-peak-width Γ_w and $k_B\mathcal{T}$. Also, in a single

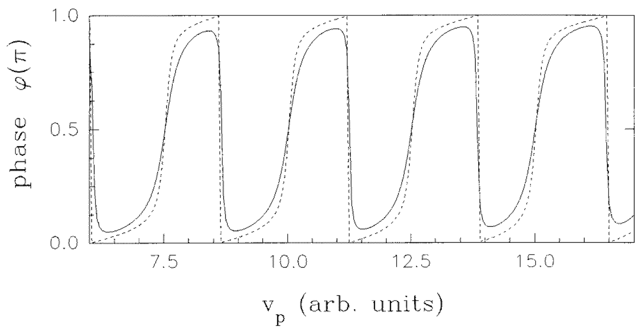

Figure 5. The phase φ versus v_g for $\mathcal{T} \neq 0$, with $k_B\mathcal{T} = 0.2$. The other parameters are the same as for figure 2. The dotted curve corresponds to the case where $\mathcal{T} = 0$, and is given for comparison.

resonance peak the phase slightly increases, slowly, and the resonance peak becomes a little wider.

It should be pointed out that in the above numerical calculation we have neglected the intra-dot Coulomb interaction (by setting $U = 0$)—not only for simplicity, but also to check whether this abrupt phase drop is a single-electron effect. In fact, if the interaction is included, the results will be qualitatively the same, and, in particular, the abrupt phase drop, by π, will still occur.

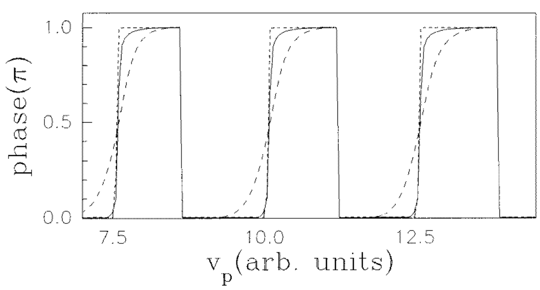

Figure 6. The phase φ versus v_p for different values of Γ_0^2 and Γ_0^3, with $\Gamma_0^1 = \Gamma_0^4 = 100$, and $\epsilon_0 = 500$. The dotted, solid, and dashed curves correspond to $\Gamma_0^2 = \Gamma_0^3 = 2$, 50, and 1000, respectively. The other parameters are the same as for figure 2.

4. The crossover of the phase behaviour on going from a four-terminal system to a two-terminal system

In this section, we turn to studying the crossover of the phase behaviour in going from the four-terminal system to a two-terminal system. It is well know that for a two-terminal phase-coherent system, the phase of the transmission amplitude can only take two values (either 0 or π), and no continuous phase variation occurs. This had been predicted theoretically about ten years ago by Büttiker on the basis of time-reversal invariance and current conservation [2]. Recently Yacoby *et al* [1] demonstrated this behaviour by using a modified Aharonov–Bohm ring, and renewed the interest of the theorists [3–6]. Here, on the basis of our theoretical result, we can show that the crossover from a continuous phase variation for the four-terminal system to an abrupt phase variation for the two-terminal system is induced just by changing the parameters. Notice that in the two-terminal experiment by Yacoby *et al* [1], they measure the current versus the magnetic flux Φ at small bias; here, the phase shift φ studied is the phase of the lowest-order harmonic wave of the collector conductance $\mathrm{d}I_4/\mathrm{d}v_1$ versus the magnetic flux Φ, with $\varphi(v_p = -\infty)$ set at zero. Figure 6 shows the dependence of the phase φ on the gate voltage v_p for different linewidths Γ_0^2, Γ_0^3. If both Γ_0^2 and Γ_0^3 are large, the phase will rise continuously, by a total of π, on an energy scale of about the half-peak-width in a resonance peak. With the decreasing of Γ_0^2 and Γ_0^3, the couplings between the bases (lead 2 and lead 3) and the dot 0 become more and more weakened, and the continuous rise becomes more and more steep. In the limit of $\Gamma_0^2 = \Gamma_0^3 = 0$, the four-terminal system reduces to a two-terminal system, and the phase variation behaves as follows.

(1) The phase abruptly rises by π near the top of a resonance peak.

(2) The phase abruptly drops by π near the halfway point between two consecutive resonance peaks.

(3) As a result of (1) and (2), the corresponding points of the successive peaks are in phase.

These theoretical results are well consistent with the experiment of Yacoby *et al* [4]. Moreover, with the decrease of Γ_0^2 and Γ_0^3, the resonance peaks of the collector conductance $\mathrm{d}I_4/\mathrm{d}v_1$ versus the gate voltage v_p change only slightly (not shown here).

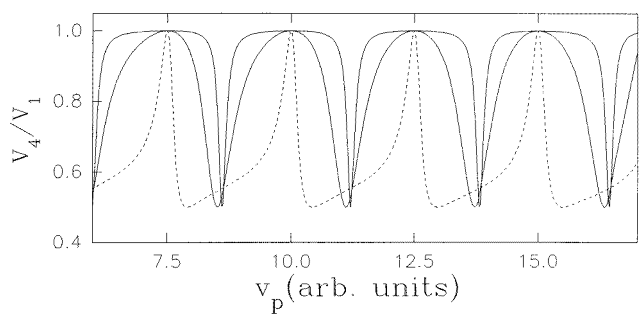

Figure 7. v_4/v_1 versus v_p for $\Gamma_1 > \Delta\epsilon$. The two solid curves correspond to $\Gamma_{ii}^1 = \Gamma_{ii}^4 = 5$ and $\Gamma_{ii}^1 = \Gamma_{ii}^4 = 20$, respectively. The dotted curve corresponds to $\Gamma_{ii}^1 = \Gamma_{ii}^4 = 0.5$. Here, $\Phi = 0$, and the other parameters are the same as for figure 2.

5. Another manifestation of off-diagonal linewidths

Taking into consideration the off-diagonal linewidths Γ_{ij} may also lead to some other interesting predictions. Here we present one of the predictions for the four-terminal phase-coherent system. The solid lines in figure 7 show the dependence of v_4/v_1 on the gate voltage v_p for the strong-coupling case, i.e. where the linewidth Γ_{ii} is larger than the interval between two peaks (the dotted line for the weak-coupling case, i.e. small Γ_{ii}, is presented for comparison). The characteristic feature is a valley that appears near the halfway point between two consecutive peaks. If we neglected the off-diagonal linewidth, v_4/v_1 would approach 1 at all values of v_p for large Γ_{ii}, due to the energy level broadening. However, near the abrupt-drop point of the phase, the off-diagonal linewidth produces so strong an influence that a valley is obtained.

6. Conclusions

In summary, a four-terminal phase-coherent system is studied to mimic the experiment of Schuster *et al*. Our theoretical result is in good qualitative agreement with their experiment. In particular, we have proposed a mechanism for the abrupt phase drop; we have attributed it to an off-diagonal linewidth, which is a single-electron effect. In addition, the crossover of the phase behaviour from a continuous phase rise for the four-terminal system to an abrupt phase rise for the two-terminal system is studied. Finally, a possible manifestation of off-diagonal linewidths is predicted and discussed.

Acknowledgments

The authors acknowledge helpful discussions with Mu Gao. This work was supported by the National Natural Science Foundation of China and the Doctoral Programme Foundation of the Institution of Higher Education of China.

References

[1] Yacoby A, Heiblum M, Mahalu D and Shtrikman H 1995 *Phys. Rev. Lett.* **74** 4047
[2] Büttiker M 1986 *Phys. Rev. Lett.* **57** 1761
[3] Hackenbroich G and Weidenmüller H A 1996 *Phys. Rev. Lett.* **76** 110
[4] Hackenbroich G and Weidenmüller H A 1996 *Phys. Rev.* B **53** 16 379
[5] Bruder C, Fazio R and Schoeller H 1996 *Phys. Rev. Lett.* **76** 114
[6] Levy Y A and Büttiker M 1995 *Phys. Rev.* B **52** 14 360
[7] Schuster R, Buks E, Heiblum M, Mahalu D, Umansky V and Shtrikman H 1997 *Nature* **385** 417
[8] Breit G and Wigner E 1936 *Phys. Rev.* **49** 519
[9] Oreg Y and Gefen Y 1997 *Phys. Rev.* B **55** 13 726
[10] Wingreen N S, Antti-Pekka J and Meir Y 1993 *Phys. Rev.* B **48** 8487
[11] Antti-Pekka J, Wingreen N S and Meir Y 1994 *Phys. Rev.* B **50** 5528
[12] Meir Y, Wingreen N S and Lee P A 1991 *Phys. Rev. Lett.* **66** 3048
[13] Wang L, Zhang J K and Bishop A R 1994 *Phys. Rev. Lett.* **73** 585
[14] Sun Qing-feng and Lin Tsung-han 1997 *J. Phys.: Condens. Matter* **9** 4875
[15] Wingreen N S, Jacobsen K W and Wilkins J W 1989 *Phys. Rev.* B **40** 11 834